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Wave propagation in a piezoelectric rod of 6 mm symmetry

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Abstract

The wave propagation in a piezoelectric rod of 6 mm symmetry is investigated by applying a 3-D piezoelectric elastic model. A self-adjoint method is introduced to solve this problem, this method avoids calculating the generalized eigenvalue equation, it completely draws the dispersion curves in the forms of Quasi-P wave, Quasi-SV wave and Quasi-SH wave under the self-adjoint boundary condition, and it can evaluate the dispersion curves of all kinds of boundary conditions. As an example, the dispersion curves of PLT-5H are completely drawn, we also found the Quasi-SV wave has standing wave phenomenon in the PLT-5H rod. In addition the relation of dispersion curves among different boundary conditions is discussed, and an experiment method is introduce to decide the dispersion curves for another boundary conditions.

Keywords: Self-adjoint method; Piezoelectric rod; Dispersion curves

1. Introduction

Piezoelectric materials (PEM) are used to fabricate sensors for various applications; they are always in layer or film forms using surface waves to realize the energy exchange with their environment. In this paper, the wave propagation in piezoelectric rod of 6 mm symmetry is investigated by applying a 3-D piezoelectric elastic model, in order to realize the energy exchange by bulk waves, for they have the low dissipative attenuation and the good oriented capacity. Furthermore the PEM can be fabricated in three dimensions, the bulk waves and surface waves can be applied together to actualize the information exchange with its environment.

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For the FEM always in layer or film forms, the study of surface waves in PEM is very abundant. The lamb wave propagation in thin piezoelectric layer bonded an elastic substrata or the thin piezoelectric layer, which is sandwiched by to semi-infinite body are studied (Joshi and Jin, 1991; Mesquida et al., 1998; Jin et al., 2002). The love wave propagation in thin piezoelectric layer bonded an elastic substrata is considered (Liu et al., 2001). Seshadri (1991) takes a method to check the corrugated surface by using Rayleigh wave. Darinkii and Weihnacht (2002) study a fast surface wave in piezoelectric layer. Mayer (1995) studies the surface acoustic waves in non-linear elastic media. But in the way of study the bulk waves in the PEM is little. Paul and Venkatesan (1987, 1988) study the vibration problems of arbitrary cross section using Fourier expansion collocation method. Guzellsu and Saha (1981) investigate the wave propagation in bones that belong to hexagonal, but they ignore the influence of electric fields in stresses. Wislon and Morrison (1977) study the waves propagation in piezoelectric rods of hexagonal crystal. Paul and Venkatesan (1989) study the waves propagation problems of arbitrary cross section using Fourier expansion collocation method. In these papers, the authors all get the general solution and derive the dispersion equation for some special boundary conditions, but their dispersion equations are depended on the roots of a generalized eigenvalue equation, so it is very hard to get the dispersion curves, and this method is very difficult to evaluate the dispersion curves of another boundary conditions. A general solution of 3-D problem is obtained by using differential operator method (Wang and Zheng, 1995). Ding et al. (2002, 2003) consider the wave, which propagates along radial direction in a piezoelectric cylinder in the case of axial symmetry and plane strain.

This paper is an attempt on the bulk waves propagation in an endless piezoelectric rod (of hexagonal crystal 6 mm). A self-adjoint method is introduced to solve this problem, this method avoids calculating the generalized eigenvalue equation, which is mentioned above, it completely draws the dispersion curves in the forms of Quasi-P wave, Quasi-SV wave and Quasi-SH wave under the self-adjoint boundary condition, and it can evaluate the dispersion curves of all kinds of boundary conditions. A finite integral is applied to expand the radial part of the wave in orthogonal completeness base and combine the governing equations and the boundary condition together (to realize the self-adjoint), after two group orthogonal completeness bases are found, and their corresponding self-adjoint boundary conditions are obtained too, the dispersion equations and guided wave conditions are derived, respectively. As an example the dispersion curves, phase velocity curves, group velocity curves and vibration amplitude along radial direction of PLT-5H are completely drawn and compared under these two group special boundary conditions; we also found the Quasi-SV wave has standing wave phenomenon in the PLT-5H rod. Although the dispersion equations are not directly obtained for another boundary conditions, quoted the theory on the relation of the body vibration frequencies with different boundary conditions being studied by Gladwell (1986), the dispersion curves for different boundary conditions could be drawn out by some aided experiments.

The contents in this paper are arranged in following sequence. First, the governing equations for axial symmetry waves in piezoelectric rod (6 mm) are discussed in Section 2. In Section 3, the finite integral is done, the radial part of the wave is expanded in orthogonal completeness base (the detail will discuss in Appendix A), the governing equations and the boundary condition are combined together (to realize the self-adjoint). Then applying two group self-adjoint boundary conditions, the corresponding guided wave conditions and dispersion equations are derived. In Section 4, the numerical calculation is performed; the frequency dispersion curves, phase velocity curves, group velocity curves and the vibration amplitude along radial directions are completely drawn under the two group special boundary conditions. Section 5 is our conclusion. In Appendix A, the two group orthogonal completeness bases are discussed in detail. In Appendix B, the relation of dispersion curves of all different boundary conditions is discussed, and an experiment method is introduced to draw the dispersion curves for another boundary conditions.

2. Governing equations

We consider a linear piezoelectric body that is a cylinder in the Cartesian frame. The equations of motion and Gauss' law have the following representation:

$$\nabla \cdot \mathbf{T} = \rho \ddot{\mathbf{u}} \tag{1}$$

$$\nabla \cdot \mathbf{D} = 0 \tag{2}$$

where, $\mathbf{T}$, $\mathbf{u}$, $\mathbf{D}$ and ρ are denoted the stress tensor, the displacement vector, the dielectric displacement vector and the mass density, respectively. A superimposed dot indicates differentiation with respect to the time parameter t, and the symbol $\nabla\cdot$ represents divergence with respect to the spatial coordinates $\mathbf{x}$. Here both body forces and body charges are neglected for simplicity.

The linear constitutive relations for piezoelectric solids are in following:

$$\mathbf{T} = \mathbf{cS} - \mathbf{eE} \tag{3}$$

$$\mathbf{D} = \mathbf{e}^{\mathrm{T}}\mathbf{S} + \boldsymbol{\varepsilon}\mathbf{E} \tag{4}$$

where, $\mathbf{c}$, $\mathbf{e}$ and $\boldsymbol{\varepsilon}$ are the elasticity tensor, the piezoelectric tensor and the dielectric tensor, respectively. A superimposed $\mathbf{T}$ indicates transpose. The strain tensor $\mathbf{S}$ and the electric field intensity vector $\mathbf{E}$ are related to the displacement vector $\mathbf{u}$ and the electric potential ϕ through the following:

$$\mathbf{S} = \frac{\mathbf{1}}{\mathbf{2}}[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}] \tag{5}$$

$$\mathbf{E} = -\nabla\phi \tag{6}$$

In the above equations, the symbol ∇ indicates gradient.

We consider PEM of a hexagonal crystal (6 mm), and notice that polycrystalline ferroelectric's ceramics are of the same symmetry. Placing the z-axis along the sixfold axis and using the compressed matrix notation, we represent the elasticity tensor, the piezoelectric tensor and the dielectric tensor by the following matrices:

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \tag{7}$$

where, $c_{66} = (c_{11} - c_{12})/2$. In the cylinder coordinate the equations of motion (1) and Gauss' law (2) take the following forms, we consider the axial-symmetric case:

$$\begin{aligned}
&c_{11}\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{1}{r^2}u_r\right) + c_{44}\frac{\partial^2 u_r}{\partial z^2} + (c_{13} + c_{44})\frac{\partial^2 u_z}{\partial r \partial z} + (e_{15} + e_{31})\frac{\partial^2 \phi}{\partial r \partial z} = \rho\frac{\partial^2 u_r}{\partial t^2} \\
&c_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{1}{r^2}u_\theta\right) + c_{44}\frac{\partial^2 u_\theta}{\partial z^2} = \rho\frac{\partial^2 u_\theta}{\partial t^2} \\
&(c_{13} + c_{44})\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) + c_{44}\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) + c_{33}\frac{\partial^2 u_z}{\partial z^2} + e_{15}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) + e_{33}\frac{\partial^2 \phi}{\partial z^2} = \rho\frac{\partial^2 u_z}{\partial t^2} \\
&(e_{15} + e_{31})\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) + e_{15}\left(\frac{\partial^2 u_r}{\partial r \partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\right) + e_{33}\frac{\partial^2 u_z}{\partial z^2} - \varepsilon_{11}\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r}\right) - \varepsilon_{33}\frac{\partial^2 \phi}{\partial z^2} = 0
\end{aligned} \tag{8}$$

The following non-dimensional forms are introduced:

$$\bar{r} = \frac{r}{b}, \quad \bar{z} = \frac{z}{b}, \quad \bar{t} = \sqrt{\frac{c_{33}}{\rho}}\frac{t}{b}, \quad u = \frac{u_r}{b}, \quad v = \frac{u_\theta}{b}, \quad w = \frac{u_z}{b}, \quad \varphi = \sqrt{\frac{\varepsilon_{33}}{c_{33}}}\frac{\phi}{b}$$

$$c_1 = \frac{c_{11}}{c_{33}}, \quad c_2 = \frac{c_{12}}{c_{33}}, \quad c_3 = \frac{c_{13}}{c_{33}}, \quad c_4 = \frac{c_{44}}{c_{33}}, \quad c_5 = \frac{c_{66}}{c_{33}}$$

$$e_1 = \frac{e_{15}}{\sqrt{c_{33}\varepsilon_{33}}}, \quad e_2 = \frac{e_{31}}{\sqrt{c_{33}\varepsilon_{33}}}, \quad e_3 = \frac{e_{33}}{\sqrt{c_{33}\varepsilon_{33}}}, \quad \varepsilon_1 = \frac{\varepsilon_{11}}{\varepsilon_{33}}$$

$$\sigma_1 = \frac{\sigma_{rr}}{c_{33}}, \quad \sigma_2 = \frac{\sigma_{\theta\theta}}{c_{33}}, \quad \sigma_3 = \frac{\sigma_{zz}}{c_{33}}, \quad \sigma_4 = \frac{\sigma_{\theta z}}{c_{33}}, \quad \sigma_5 = \frac{\sigma_{zr}}{c_{33}}, \quad \sigma_6 = \frac{\sigma_{r\theta}}{c_{33}}$$

$$D_1 = \frac{D_r}{\sqrt{c_{33}\varepsilon_{33}}}, \quad D_2 = \frac{D_\theta}{\sqrt{c_{33}\varepsilon_{33}}}, \quad D_3 = \frac{D_z}{\sqrt{c_{33}\varepsilon_{33}}}$$

where, b is the outer radius of the cylinder. Then rewrite Eq. (8), yield:

$$c_1\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{r^2}u\right) + c_4\frac{\partial^2 u}{\partial z^2} + (c_3 + c_4)\frac{\partial^2 w}{\partial r \partial z} + (e_1 + e_2)\frac{\partial^2 \varphi}{\partial r \partial z} = \frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$c_5\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}v\right) + c_4\frac{\partial^2 v}{\partial z^2} = \frac{\partial^2 v}{\partial t^2} \tag{10}$$

$$(c_3 + c_4)\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + c_4\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{\partial^2 w}{\partial z^2} + e_1\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) + e_3\frac{\partial^2 \varphi}{\partial z^2} = \frac{\partial^2 w}{\partial t^2} \tag{11}$$

$$(e_1 + e_2)\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + e_1\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + e_3\frac{\partial^2 w}{\partial z^2} - \varepsilon_1\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) - \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{12}$$

For convenience, we still use r, z and t in Eqs. (9)–(12).

A finite integral method is introduced to assist in solving this problem.

3. The solution of equations

First, the finite Bessel integral is used to expand the radial part of the wave propagation in the rod. The finite Bessel integral is:

$$f^{J_v}(\mu) = \int_0^1 f(r)J_v(\mu r)r\,\mathrm{d}r, \quad v = 0, 1, \tag{13}$$

where, $J_v(\mu r)$ is the first kind of Bessel function, μ is the integral variable. $v = 0,1$, means that only the zero class and first class Bessel functions are used. Apply the first class of finite Bessel integral to the Eqs. (9) and (10), and the zero class of finite Bessel integral to the Eqs. (11) and (12), yield in matrix forms:

$$\begin{bmatrix} c_1\mu^2 - c_4\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial t^2} & (c_3+c_4)\mu\frac{\partial}{\partial z} & (e_1+e_2)\mu\frac{\partial}{\partial z} \\ (c_3+c_4)\mu\frac{\partial}{\partial z} & \frac{\partial^2}{\partial z^2} - c_4\mu^2 - \frac{\partial^2}{\partial t^2} & -e_1\mu^2 + e_3\frac{\partial^2}{\partial z^2} \\ (e_1+e_2)\mu\frac{\partial}{\partial z} & -e_1\mu^2 + e_3\frac{\partial^2}{\partial z^2} & \varepsilon_1\mu^2 - \frac{\partial^2}{\partial z^2} \end{bmatrix} \begin{bmatrix} u^{J_1} \\ w^{J_0} \\ \varphi^{J_0} \end{bmatrix}$$

$$= \begin{bmatrix} c_1J_1(\mu r)u + c_1J_1(\mu r)r\frac{\partial u}{\partial r} - c_1\mu rJ_0(\mu r)u + (c_3+c_4)J_1(\mu r)r\frac{\partial w}{\partial z} + (e_1+e_2)J_1(\mu r)r\frac{\partial \varphi}{\partial z} \\ -(c_3+c_4)rJ_0(\mu r)\frac{\partial u}{\partial z} - c_4J_0(\mu r)r\frac{\partial w}{\partial r} - c_4\mu rJ_1(\mu r)w - e_1J_0(\mu r)r\frac{\partial \varphi}{\partial r} - e_1\mu rJ_1(\mu r)\varphi \\ -e_1J_0(\mu r)r\frac{\partial w}{\partial r} - e_1\mu rJ_1(\mu r)w - (e_1+e_2)rJ_0(\mu r)\frac{\partial u}{\partial z} + \varepsilon_1 rJ_0(\mu r)\frac{\partial \varphi}{\partial r} + \varepsilon_1\mu rJ_1(\mu r)\varphi \end{bmatrix}_{r=1}$$

$$\times\, c_5\mu^2 v^{J_1} - c_4\frac{\partial^2 v^{J_1}}{\partial z^2} + \frac{\partial^2 v^{J_1}}{\partial t^2} = \left[c_5J_1(\mu r)v + c_5J_1(\mu r)r\frac{\partial v}{\partial r} - c_5\mu rJ_0(\mu r)v\right]_{r=1} \quad (14a,b)$$

So Eq. (14a,b), which combine the governing equations and the boundary information together, are the dynamics equations in domain (μ:z, t). Eq. (14a) is about displacement u, w and electric potential φ; Eq. (14b) is only about displacement v. So the Quasi-SH wave could propagate solely without the influence on electric potential.

Let us assume that the wave propagation in the rod have following forms:

$$\begin{aligned} &u(r,z,t) = f_1(r)e^{i(kz-\omega t)}, \quad v(r,z,t) = f_2(r)e^{i(kz-\omega t)} \\ &w(r,z,t) = f_3(r)e^{i(kz-\omega t)}, \quad \phi(r,z,t) = f_4(r)e^{i(kz-\omega t)} \end{aligned} \quad (15)$$

where, $f_j(r), j = 1,2,3,4$ are the wave propagation amplitude coefficients, k is the wave number, ω is the circular frequency. So the displacements and electric potential u^{J_1}, v^{J_1}, w^{J_0}, φ^{J_0} can be written in following forms:

$$\begin{aligned} &u^{J_1} = A_1(\mu)e^{i(kz-\omega t)}, \quad v^{J_1} = A_2(\mu)e^{i(kz-\omega t)} \\ &w^{J_0} = A_3(\mu)e^{i(kz-\omega t)}, \quad \varphi^{J_0} = A_4(\mu)e^{i(kz-\omega t)} \end{aligned} \quad (16)$$

where, $A_j(\mu) = \int_0^1 f_j(r)J_v(\mu r)r\,dr, j = 1,2,3,4$, and while $j = 1,2$, $v = 1$, while $j = 3,4$, $v = 0$. Substitute Eq. (16) into the left parts of Eqs. (14a,b), give:

$$\begin{bmatrix} c_1\mu^2 + c_4k^2 - \omega^2 & i(c_3+c_4)\mu k & i(e_1+e_2)\mu k \\ i(c_3+c_4)\mu k & -k^2 - c_4\mu^2 + \omega^2 & -e_1\mu^2 - e_3k^2 \\ i(e_1+e_2)\mu k & -e_1\mu^2 - e_3k^2 & \varepsilon_1\mu^2 + k^2 \end{bmatrix} \begin{bmatrix} A_1(\mu)e^{i(kz-\omega t)} \\ A_3(\mu)e^{i(kz-\omega t)} \\ A_4(\mu)e^{i(kz-\omega t)} \end{bmatrix}$$

$$= \begin{bmatrix} c_1J_1(\mu r)u + c_1J_1(\mu r)r\frac{\partial u}{\partial r} - c_1\mu rJ_0(\mu r)u + (c_3+c_4)J_1(\mu r)r\frac{\partial w}{\partial z} + (e_1+e_2)J_1(\mu r)r\frac{\partial \varphi}{\partial z} \\ -(c_3+c_4)rJ_0(\mu r)\frac{\partial u}{\partial z} - c_4J_0(\mu r)r\frac{\partial w}{\partial r} - c_4\mu rJ_1(\mu r)w - e_1J_0(\mu r)r\frac{\partial \varphi}{\partial r} - e_1\mu rJ_1(\mu r)\varphi \\ -e_1J_0(\mu r)r\frac{\partial w}{\partial r} - e_1\mu rJ_1(\mu r)w - (e_1+e_2)rJ_0(\mu r)\frac{\partial u}{\partial z} + \varepsilon_1 rJ_0(\mu r)\frac{\partial \varphi}{\partial r} + \varepsilon_1\mu rJ_1(\mu r)\varphi \end{bmatrix}_{r=1}$$

$$\times\, (c_5\mu^2 + c_4k^2 - \omega^2)A_2(\mu)e^{i(kz-\omega t)} = \left[c_5J_1(\mu r)v + c_5J_1(\mu r)r\frac{\partial v}{\partial r} - c_5\mu rJ_0(\mu r)v\right]_{r=1} \quad (17a,b)$$

So Eq. (17a,b) are about the parameters (μ, k, ω). When we choice the appropriate boundary condition, if which makes the right part of Eqs. (17a,b) becoming zero and calculates the value of μ, the dispersion equation can be derived, furthermore, the dispersion curves under such boundary condition are obtained.

We consider two group self-adjoint boundary conditions for the rod,

Elastic simply supported (BI):

$$\begin{aligned} &w = 0, \quad r\sigma_1 + 2c_5u = 0, \quad r\sigma_6 + 2c_5v = 0, \qquad \text{(Displacements and stress conditions)} \\ &\varphi = 0, \quad \text{(Electric field condition)} \end{aligned} \quad (18)$$

and

Rigid sliding supported (BII):

$$\begin{aligned} &u = 0, \quad v = 0, \quad \sigma_5 = 0, \qquad \text{(Displacements and stress conditions)} \\ &D_1 = 0, \quad \text{(Electric field condition)} \end{aligned} \tag{19}$$

Substitute the two group boundary conditions (18) and (19) into the right parts of Eq. (17a,b), the right parts of Eq. (17a,b) will equal to zero, respectively. And give:

$$\begin{bmatrix} c_1\mu^2 + c_4k^2 - \omega^2 & i(c_3 + c_4)\mu k & i(e_1 + e_2)\mu k \\ i(c_3 + c_4)\mu k & -k^2 - c_4\mu^2 + \omega^2 & -e_1\mu^2 - e_3k^2 \\ i(e_1 + e_2)\mu k & -e_1\mu^2 - e_3k^2 & \varepsilon_1\mu^2 + k^2 \end{bmatrix} \begin{bmatrix} A_1(\mu) \\ A_3(\mu) \\ A_4(\mu) \end{bmatrix} = 0 \tag{20a}$$

$$(c_5\mu^2 + c_4k^2 - \omega^2)A_2(\mu) = 0. \tag{20b}$$

where, the value of μ is decided by guided wave condition

$$J_0(\mu) = 0, \quad \text{(Boundary condition BI)} \tag{21}$$

and

$$J_1(\mu) = 0, \quad \text{(Boundary condition BII)} \tag{22}$$

When the waves propagate in the rod, the Eq. (20a,b) have the non-zero solutions, which need the determinants of the coefficients matrices of Eq. (20a,b) equal to zero:

$$\det \begin{pmatrix} c_1\mu^2 + c_4k^2 - \omega^2 & i(c_3 + c_4)\mu k & i(e_1 + e_2)\mu k \\ i(c_3 + c_4)\mu k & -k^2 - c_4\mu^2 + \omega^2 & -e_1\mu^2 - e_3k^2 \\ i(e_1 + e_2)\mu k & -e_1\mu^2 - e_3k^2 & \varepsilon_1\mu^2 + k^2 \end{pmatrix} = 0 \tag{23a}$$

$$\det(c_5\mu^2 + c_4k^2 - \omega^2) = 0 \tag{23b}$$

Formulas (23a,b) are the frequency dispersion equations of the two group boundary conditions (BI and BII).

The roots of $J_0(\mu) = 0$ (BI) or the roots of $J_1(\mu) = 0$ (BII) are a sequence, noted μ_{in}, $n = 1, 2, 3, \ldots$, $i = 0, 1$.

4. Numerical calculation

In this section, the numerical calculation is performed to get the frequency dispersion curves, phase velocity curves, group velocity curves and the vibration amplitude along radial direction under boundary conditions BI and BII. For calculating, the data of PLT-5H ferroelectric's ceramics are introduced in following, $c_1 = 121/117$, $c_2 = 53/78$, $c_3 = 841/1170$, $c_4 = 23/117$, $c_5 = 83/468$, $e_1 = 17/39$, $e_2 = -1/6$, $e_3 = 233/390$, $\varepsilon_1 = 15/13$.

Fig. 1 shows the frequency dispersion curves of boundary condition BI. There have three curves for every number (1, 2 or 3); they are Quasi-P wave, Quasi-SV wave and Quasi-SH wave. The symbols P, SV and SH present Quasi-P wave, Quasi-SV wave and Quasi-SH wave. The numbers of 1, 2 and 3 are corresponding to the values of μ_{0n}, $n = 1, 2, 3$. The two dash-dot lines are the asymptotes; the P line is the asymptote of Quasi-P wave, The S line is the asymptote of Quasi-SV wave and Quasi-SH wave.

Fig. 2 shows the frequency dispersion curves of Quasi-P wave under boundary condition BI and BII. The dispersion curve of boundary condition BII is higher than the dispersion curve of boundary condition BI

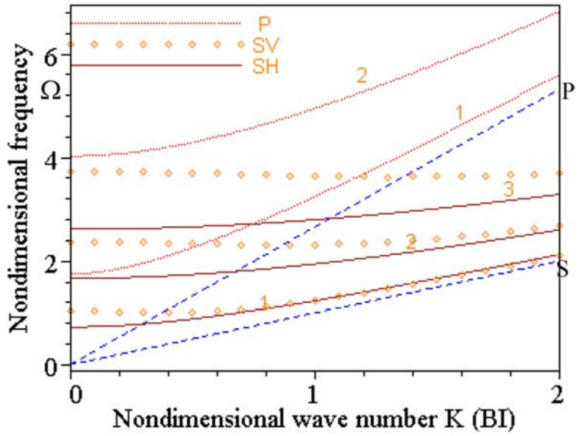

Fig. 1. Frequency dispersion curves (BI).

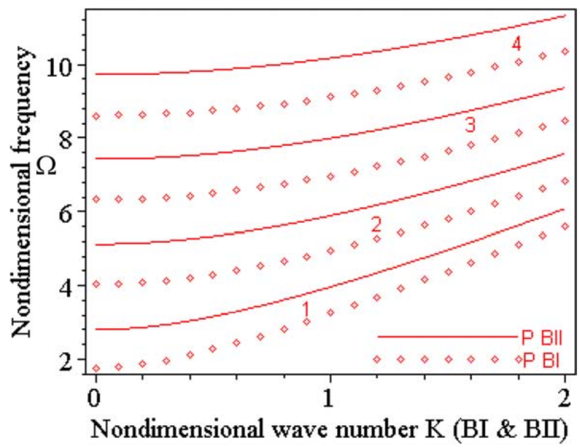

Fig. 2. Frequency dispersion curves (BI & BII Quasi-P wave).

for every number (1, 2, 3 or 4). It implies that the frequency dispersion curves of Quasi-P wave for different boundary conditions have some regularity. This detail will discuss in Appendix B.

Fig. 3 shows the frequency dispersion curves of Quasi-SV wave and Quasi-SH wave under boundary condition BI and BII. The dispersion curve of boundary condition BII is higher than the dispersion curve of boundary condition BI for every number (1, 2 or 3). It implies that the frequency dispersion curves of Quasi-SV wave and Quasi-SH for different boundary conditions have some regularity. This detail will discuss in Appendix B.

Fig. 4 shows the phase velocity curves of boundary condition BI. The dash-dot lines of P and S present the velocity $c_P = \sqrt{(c_{33} + 2c_{44})/\rho}$ and $c_S = \sqrt{c_{44}/\rho}$.

Fig. 5 shows the group velocity curves of boundary condition BI. The Quasi-SV wave has standing wave phenomenon in PLT-5H material.

Figs. 6 and 7 show the vibration amplitude of along radial direction curves of boundary condition BI and BII.

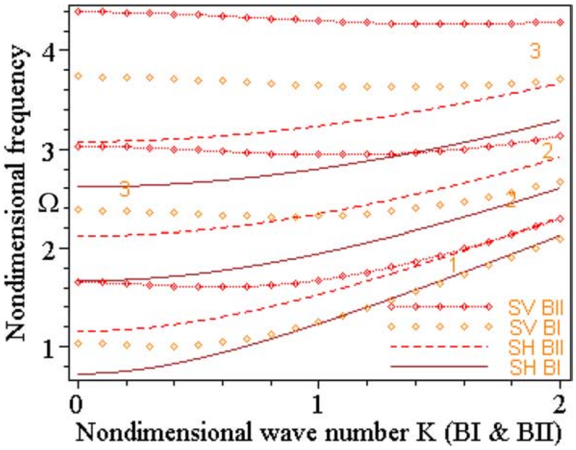

Fig. 3. Frequency dispersion curves (BI & BII Quasi-SV and Quasi-SH waves).

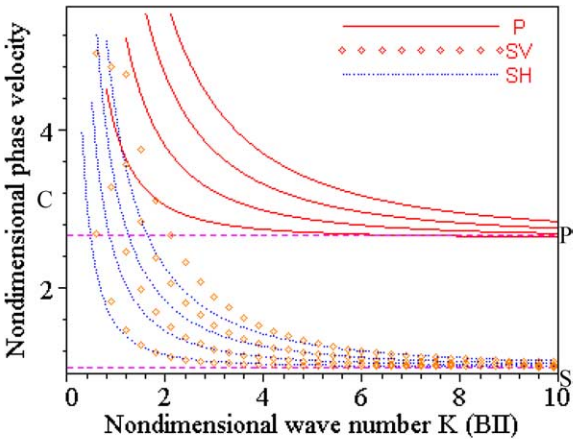

Fig. 4. Phase velocity curves (BII).

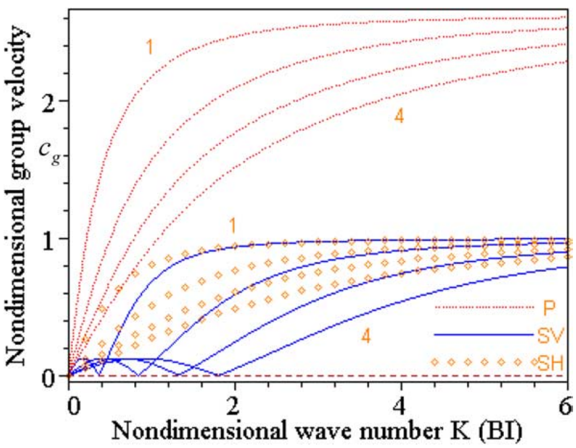

Fig. 5. Group velocity curves (BI).

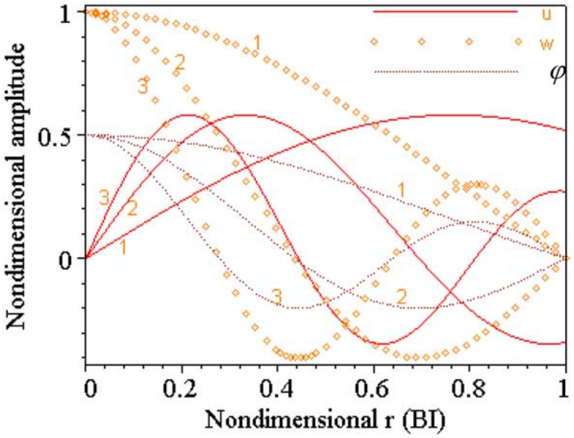

Fig. 6. Vibration amplitude (BI).

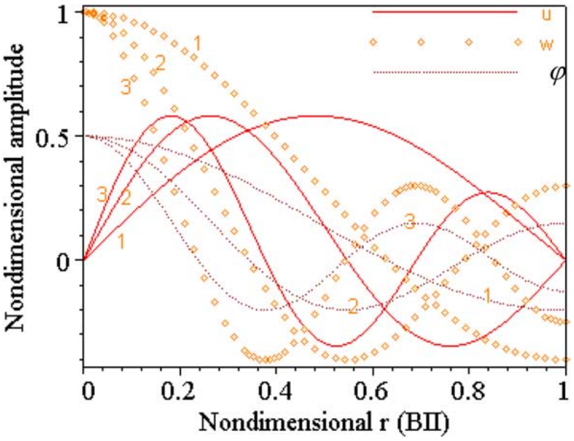

Fig. 7. Vibration amplitude (BII).

5. Conclusion

The wave propagation in a piezoelectric rod of 6 mm symmetry is solved. The dispersion curves are plotted in the forms of Quasi-P wave, Quasi-SV wave and Quasi-SH wave. As an example the dispersion curves of PLT-5H are completely drawn, we also found the Quasi-SV wave has standing wave phenomenon in the PLT-5H rod.

Elastic waves provide a useful and versatile way of non-destructively testing various structures. We hope to apply all kinds waves, not only the surface waves, also the bulk waves in PEM, to actualize the non-destructively testing, also to ascertain the damage position and damage degree for various structures in the future.

Although the symmetrical problem is studied in the paper, the asymmetrical problem will be studied following.

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Appendix A

Let us assume that μ_{0i} and μ_{0j} are two arbitrary roots of $J_0(\mu) = 0$, do the following integral, gives:

$$\int_0^1 J_0(\mu_{0i}r)J_0(\mu_{0j}r)r\,\mathrm{d}r = \begin{cases} 0 & i \neq j \\ 0.5J_1^2(\mu_{0i}) & i = j' \end{cases}$$

similarly, if μ_{1i} and μ_{1j} are two arbitrary roots of $J_1(\mu) = 0$, get:

$$\int_0^1 J_1(\mu_{1i}r)J_1(\mu_{1j}r)r\,\mathrm{d}r = \begin{cases} 0 & i \neq j \\ -0.5J_0(\mu_{1i})J_2(\mu_{1i}) & i = j \end{cases}$$

We could see that the sequences of $\{J_0(\mu_{0n}r)|r \in [0,1]\}$ and $\{J_1(\mu_{1n}r)|r \in [0,1]\}$ are both orthogonal completeness sequences. It is clear that for arbitrary function $f(r)$ could be written in series:

$$f(r) = \sum_{n=1}^{\infty} A_n(\mu_n)J_v(\mu_n r), \quad v = 0, 1$$

where,

$$A_n(\mu_n) = \int_0^1 f(r)J_v(\mu_n r)r\,\mathrm{d}r, \quad v = 0, 1$$

So that the radial part of the wave propagation is expanded into orthogonal modes, and the finite integral realizes the self-adjoint of governing equations with boundary conditions.

Appendix B

In this paper, although only two group orthogonal completeness bases and their self-adjoint boundary conditions are found and the dispersion curves are completely drawn only for these two group boundary conditions, the dispersion curves can be outlined for another different boundary conditions (it is very difficult to get the orthogonal completeness base for every boundary condition).

For a structure, if the boundary conditions are changed, the frequencies of the body will be changed too, but the change has the regularity by Gladwell (1986). Assuming $\omega_{jm}, j, m = 1, 2, 3, \ldots$ are the frequencies; j is for different boundary conditions, m is the mth mode, so they have following inequality:

$$0 < \omega_{11} < \omega_{21} < \omega_{31} < \cdots < \omega_{12} < \omega_{22} < \omega_{32} < \cdots \omega_{1m} < \omega_{2m} < \omega_{3m} < \cdots \tag{B.1}$$

It also can be seen in Figs. 2 and 3 that the same kind of curves (just as Quasi-P wave) of BII are higher than the curves of BI for the same n and the same kind of curves don't cross, and there has the inequality $\mu_{01} < \mu_{11} < \mu_{02} < \mu_{12} < \mu_{03} < \mu_{13}$ by calculating $J_0(\mu) = 0$ and $J_1(\mu) = 0$. So, we could decide the dispersion curves for arbitrary boundary condition from some aided experiments. If we get the frequency Ω and wave number K of the curve and evaluate the curve's type (Quasi-P wave, Quasi-SV wave or Quasi-SH wave), we can mark a point in Fig. 2 or Fig. 3, after a few points we could line the dispersion curves for this boundary condition.

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